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Letter to the Editors

Note on the Paper "Boundary Conditions in Finite Difference Fluid Dynamic Codes" by C. K. Chu and Aron Sereny

As pointed out by Chu and Sereny [1], the number of reported test calculations with different numerical methods for mixed initial-boundary value problems in fluid dynamics is very small. Although the present paper only considers a single basic difference approximation, the Lax–Wendroff scheme, and a test problem (2) where all advective terms are absent, the main points and results are therefore no doubt worth a close attention. Some remarks should however be made.

First of all, Chu and Sereny [1, p. 480] call attention to the difference between actual, wave-equation-like systems and the characteristic model equation $u_t + u_x = 0$. For a single equation (or systems with only forward characteristics), the use of overspecified boundary conditions to a dissipative scheme is sometimes not serious. To a system with characteristics of both types, coupled through the physical boundary conditions, such overspecifications may be devastating. Unfortunately, this important difference between wave-equation-like systems and equations in characteristic form thus is not always observed by the authors.

(i) On page 480, the final form (5) of the Lax–Wendroff scheme for a constant-coefficient case is e.g. not correct. The last term should be

$$-(\Delta t/2)\, a^2\rho^2(F_{j+1}^n + F_{j-1}^n - 2F_j^{\,n})/(\Delta\xi)^2$$

instead of

$$-(\Delta t/2)(G_{j+1}^n + G_{j-1}^n - 2G_j^{\,n})/(\Delta\xi)^2.$$

(ii) On page 482, reference is made to stability proofs in a paper by Gustafsson, Kreiss, and Sundström: "When Eqs. (2) are first cast into diagonal (or characteristic) form, then Gustafsson *et al.* have proved the stability for schemes (2), (3), (5), and (7)." The stability properties of the present problem (with non-characteristic quantities given as boundary conditions) however can not be deduced from the results for a system with characteristic boundary conditions. The reason for this mistake may be the fact that the paper by Gustafsson *et al.* actually also contains a stability analysis for the wave equation with boundary conditions of the present type, but only for the leap-frog and Crank–Nicholson schemes. An extension of the results to the Lax–Wendroff scheme is cumbersome but not complicated (see the Appendix to this note).

A second reference to the paper by Gustafsson *et al.* is made on page 487, in an attempt to explain the "unexpected" behavior with boundary conditions (6) and (7) for problem C. The reason for the unbounded growth of ρ and u is however different and very simple. In problem C, the exact solution at the nonreflecting end should be a simple wave. This essential boundary condition is still only used in scheme (4). In all other cases, ρ and u are computed separately by different extrapolation schemes. As they stand, these schemes do not guarantee any simple-wave structure of the solution, and the observed exponential growth for boundary conditions (6) and (7) should be no more surprising than the damping obtained with conditions (3) and (5).

APPENDIX

Stability Proofs for Boundary Conditions (1)–(7) at a Solid Wall ($u = 0$)

The linearized Lax–Wendroff approximation to the system (2) in Chu and Sereny's paper is

$$V_j'^{n+1} = V_j'^{n} + \frac{\Delta t}{2\Delta\xi}\,(u_{j+1}'^{n} - u_{j-1}'^{n}) + a^2\rho^2\,\frac{\Delta t^2}{2\Delta\xi^2}\,(V_{j+1}'^{n} - 2V_j'^{n} + V_{j-1}'^{n}),$$

$$u_j'^{n+1} = u_j'^{n} + a^2\rho^2\,\frac{\Delta t}{2\Delta\xi}\,(V_{j+1}'^{n} - V_{j-1}'^{n}) + a^2\rho^2\,\frac{\Delta t^2}{2\Delta\xi^2}\,(u_{j+1}'^{n} - 2u_j'^{n} + u_{j-1}'^{n}).$$

For constant coefficients, the general solution in $l_2(x)$ with $u_0'^n = 0$ may be written

$$u_j'^n = \tfrac{1}{2}a\rho V_0'(\kappa_1^{\,j} - \kappa_2^{-j})\,z^n,$$

$$V_j'^n = \tfrac{1}{2}V_0'(\kappa_1^{\,j} + \kappa_2^{-j})\,z^n.$$

In this expression, κ_1, κ_2 are the roots of the characteristic equation

$$z\kappa = \kappa + \tfrac{1}{2}\lambda a\rho(\kappa^2 - 1) + \tfrac{1}{2}\lambda^2 a^2\rho^2(\kappa - 1)^2,$$

with $|\kappa_1| \leqslant 1$, $|\kappa_2| \geqslant 1$ for $|z| \geqslant 1$ and where $\lambda = \Delta t/\Delta\xi$.
Inserting this general expression into the different numerical boundary conditions, we get:

For boundary condition (1), $V_0'^n = 0 : V_0' = 0$; stability.

For boundary condition (2), $V_0'^n = V_1'^n$:

$$0 = \tfrac{1}{2}V_0'(\kappa_1 + \kappa_2^{-1} - 2) = \tfrac{1}{2}V_0'[\kappa_2^{-1}(1 + \kappa_1\kappa_2) - 2]$$

$$= \tfrac{1}{2}V_0'\left[\kappa_2^{-1}\left(1 - \frac{1 - \lambda a\rho}{1 + \lambda a\rho}\right) - 2\right] = V_0'\kappa_2^{-1}\left(\frac{\lambda a\rho}{1 + \lambda a\rho} - \kappa_2\right)$$

and since $0 < \lambda a\rho/(1 + \lambda a\rho) < 1$, $|\kappa_2| \geqslant 1$, this implies $V_0' = 0$; stability.

For boundary condition (3): $V_0'^n = 2V_1'^n - V_2'^n$:

$$\begin{aligned} 0 &= \tfrac{1}{2}V_0'(\kappa_1^2 + \kappa_2^{-2} - 2\kappa_1 - 2\kappa_2^{-1} + 2) \\ &= \tfrac{1}{2}V_0'[\kappa_2^{-2}(1 + \kappa_1^2\kappa_2^2) - 2\kappa_2^{-1}(1 + \kappa_1\kappa_2) + 2] \\ &= V_0'\left[\kappa_2^{-2}\frac{1 + \lambda^2a^2\rho^2}{(1 + \lambda a\rho)^2} - 2\kappa_2^{-1}\frac{\lambda a\rho}{1 + \lambda a\rho} + 1\right] \\ &= V_0'\kappa_2^{-2}\left(\kappa_2 - \frac{\lambda a\rho + i}{1 + \lambda a\rho}\right)\left(\kappa_2 - \frac{\lambda a\rho - i}{1 + \lambda a\rho}\right), \end{aligned}$$

and since $|(\lambda a\rho \pm i)/(1 + \lambda a\rho)| < 1$; $|\kappa_2| \geqslant 1$ this implies $V_0' = 0$; stability.

For boundary condition (4), $R_0'^{n+1} = (1 - \lambda a\rho)R_0'^n + \lambda R_1'^n$:

$$R_j'^n = u_j'^n + a\rho V_j'^n = a\rho V_0'\kappa_1{}^j z^n$$

gives

$$\begin{aligned} 0 &= a\rho V_0'(z - 1 - \lambda a\rho(\kappa_1 - 1)) \\ &= \lambda a^2\rho^2 V_0'[\tfrac{1}{2}(\kappa_1 - \kappa_1^{-1}) + \tfrac{1}{2}\lambda a\rho(\kappa_1 + \kappa_1^{-1} - 2) - \kappa_1 + 1] \\ &= \tfrac{1}{2}\lambda a^2\rho^2 V_0'(\lambda a\rho - 1)(\kappa_1 + \kappa_1^{-1} - 2) \\ &= \tfrac{1}{2}\lambda a^2\rho^2 V_0'(\lambda a\rho - 1)\,\kappa_1^{-1}(\kappa_1 - 1)^2. \end{aligned}$$

Since $\lambda a\rho < 1$ (stability condition for $L - W$ scheme) and $\kappa_1 \neq 1$ for all $|z| \geqslant 1$, this implies $V_0' = 0$; stability.

For boundary condition (5), $V_0'^{n+1} - V_0'^n = \lambda(u_1'^n - u_0'^n)$:

$$\begin{aligned} 0 &= V_0'(z - 1 - \tfrac{1}{2}\lambda a\rho(\kappa_1 - \kappa_2^{-1})) \\ &= \tfrac{1}{2}\lambda a\rho V_0'(\kappa_2 - \kappa_1 + \lambda a\rho(\kappa_2 + \kappa_2^{-1} - 2)) \\ &= \tfrac{1}{2}\lambda a\rho V_0'\left[\kappa_2(1 + \lambda a\rho) - 2\lambda a\rho + \left(\lambda a\rho + \frac{1 - \lambda a\rho}{1 + \lambda a\rho}\right)\kappa_2^{-1}\right] \\ &= \tfrac{1}{2}\lambda a\rho V_0'\kappa_2^{-1}(1 + \lambda a\rho)\left(\kappa_2 - \frac{\lambda a\rho + i}{1 + \lambda a\rho}\right)\left(\kappa_2 - \frac{\lambda a\rho - i}{1 + \lambda a\rho}\right), \end{aligned}$$

and in the same way as for condition (3), stability follows directly.

For boundary condition (6), $V_0'^{n+1} - V_0'^{n} = 2\lambda(u_1'^{n} - u_0'^{n}) - \frac{1}{2}\lambda(u_2'^{n} - u_0'^{n})$:

$$0 = V_0'\left(z - 1 - \lambda a\rho(\kappa_1 - \kappa_2^{-1}) + \frac{\lambda a\rho}{4}(\kappa_1^2 - \kappa_2^{-2})\right)$$

$$= V_0'\left(z - 1 - \frac{2\lambda a\rho}{1 - \lambda a\rho}\kappa_1 - \frac{\lambda^2 a^2\rho^2}{(1 - \lambda a\rho)^2}\kappa_1^2\right)$$

$$= V_0'\left(z^{1/2} - 1 - \frac{\lambda a\rho}{1 - \lambda a\rho}\kappa_1\right)\left(z^{1/2} + 1 + \frac{\lambda a\rho}{1 - \lambda a\rho}\kappa_1\right),$$

or, inserting the expression for κ_1,

$$0 = \frac{V_0'}{(1 - \lambda^2 a^2\rho^2)^2}[z^{1/2}(1 - \lambda^2 a^2\rho^2) - z \mp q]$$

$$\times [z^{1/2}(1 - \lambda^2 a^2\rho^2) + z \pm q]$$

$$= V_0'\frac{(-2z^{3/2} + (3 - \lambda^2 a^2\rho^2)z - 1)(2z^{3/2} + (3 - \lambda^2 a^2\rho^2)z - 1)}{[z^{1/2}(1 - \lambda^2 a^2\rho^2) - z \pm q] \times [z^{1/2}(1 - \lambda^2 a^2\rho^2) + z \pm q]}$$

where $q = (z^2 - (2z - 1)(1 - \lambda^2 a^2\rho^2))^{1/2}$. To prove the stability, we must only show that the equation

$$2y^3 - (3 - \lambda^2 a^2\rho^2)y^2 + 1 = 0$$

has no root with $|y|^2 = |z| \geqslant 1$. By computing the roots for suitable values of $\lambda a\rho$ in the interval $0 < \lambda a\rho < 1$, this property has been verified experimentally.

For boundary condition (7), $V_0'^{n+1} - V_0'^{n} + V_1'^{n+1} - V_1'^{n} = \lambda(u_1'^{n+1} + u_1'^{n})$:

$$0 = V_0'\left[(z - 1)\left(1 + \frac{\kappa_1 + \kappa_2^{-1}}{2}\right) - \lambda a\rho(z + 1)\frac{\kappa_1 - \kappa_2^{-1}}{2}\right]$$

$$= V_0'\left[z - 1 - 2z\frac{\lambda a\rho}{1 - \lambda a\rho}\kappa_1\right],$$

or, inserting the expression for κ_1,

$$0 = V_0'\left[z - 1 - 2z\frac{z - 1 + \lambda^2 a^2\rho^2 \pm (z^2 - (2z - 1)(1 - \lambda^2 a^2\rho^2))^{1/2}}{1 - \lambda^2 a^2\rho^2}\right]$$

$$= \frac{V_0'}{1 - \lambda^2 a^2\rho^2}[-2z^2 + 3(1 - \lambda^2 a^2\rho^2)z - (1 - \lambda^2 a^2\rho^2)$$

$$\pm 2z(z^2 - (2z - 1)(1 - \lambda^2 a^2\rho^2))^{1/2}]$$

$$= V_0'\frac{4z^3 - (3z - 1)^2(1 - \lambda^2 a^2\rho^2)}{2z^2 - (3z - 1)(1 - \lambda^2 a^2\rho^2) \pm 2z(z^2 - (2z - 1)(1 - \lambda^2 a^2\rho^2))^{1/2}}.$$

By computing the roots of $4z^3 - (3z - 1)^2 (1 - \lambda^2 a^2 \rho^2)$ for various $\lambda a \rho$ in the interval $0 < \lambda a \rho < 1$, it can be shown that for all $| z | \geqslant 1$, we must have $V_0' = 0$ and the stability is thereby shown also for the last boundary condition.

Reference

1. C. K. Chu and Aron Sereny, Boundary conditions in finite difference fluid dynamic codes, *J. Comput. Phys.* **15** (1974), 476.

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Arne Sundström
National Defence Research Institute,
Stockholm, Sweden